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The dynamics of a deformable drop suspended in an unbounded Stokes flow

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The velocity fields in and around a deformed drop suspended in an arbitrary (albeit Stokesian) unbounded flow field are solved. The usefulness of the solution is demonstrated by solving the drag force and lateral migration of a drop suspended in an unbounded Poiseuillian field.

It is demonstrated that, due to the deformation of the drop, there exists a radial component of the settling velocity. The direction of the radial migration depends primarily on the product U_{HR} (the Hadamard–Rybczynski terminal settling velocity) by U_0 (the maximum Poiseuillian velocity). A positive product results in a lateral migration away from the location of maximum velocity; the converse also holds.

1. Introduction

The motion of deformable bubbles or droplets suspended in another fluid are encountered frequently in such varied applications as boiling problems, fluidized beds, direct contact heat exchangers, etc. Blood flow is one of the more complex phenomena of medicine, the deformability of the white blood cells further complicating analysis. A rapidly increasing number of publications in the field is only one indication of the intense interest in biological systems containing blood.

The lateral migration of *solid* particles suspended in laminar flow is a well-known phenomenon which has been tested experimentally, e.g. Goldsmith & Mason (1962), Segré & Silberberg (1962); and treated analytically to some extent, e.g. Repetti & Leonard (1966) and Saffman (1965). Brenner (1966) summarized the then-existing experimental data and reviewed the theoretical attempts. In addition, he stated that a non-neutrally buoyant drop may migrate to the axis or wall of the tube, because of its deformation. This phenomenon will occur also in the Stokes flow régime. Later, Cox & Brenner (1968) advanced a theory for the lateral migration of solid particles in Poiseuille flow. That theory included the effects of the flow, the solid boundaries and the inertia terms in the Navier–Stokes equations. Due to the complexity of the formulae, no numerical computations were given and no comparison with the experimental data was made.

The motion of *deformable* bodies in a fluid is much more complex, since the shape of the interface has to be solved simultaneously with the field equations.

Bugliarello & Hsiao (1964) studied factors affecting the phenomenon of unequal distribution of blood cells in the blood vessels. Their experiments were, however, conducted with solid particles. Linegard & Whitmore (1968) stated that the complex flow properties of blood are related directly to the deformability of the leukocytes. There is other overwhelming evidence of the existence of deformation of white blood cells in flowing blood, and there have been many attempts to lump these effects in an overall property such as pseudo-plasticity. Chaffey & Brenner (1967) investigated the flow around a neutrally buoyant drop suspended in a simple shear flow, while considering the effect of the drop's deformation on the flow field. Cox (1969) solved the time-dependent flow field in and around a neutrally buoyant deformable drop submerged in an arbitrary unbounded flow. He considered, however, only first-order effects of the deformation on the flow field, and his unperturbed flow field is limited to a generalization of shear flows (Couette flow, hyperbolic flow, etc.).

Hetsroni & Haber (1970) suggested a general solution of the flow fields in and around a drop suspended in an arbitrary (but Stokesian) unbounded flow field. That solution is based on an iterative procedure, i.e. the flow fields are solved first for a spherical drop, and the geometry of the interface is then determined for the perturbed flow field. In the second iteration, presented herein, the flow field in and around the deformed drop is sought, etc.

Thus, the purpose of the present work is to provide the second iteration to our previous solution, namely to present a method of solution for the flow fields in and around a *deformed* drop† and to evaluate the drag force and the lateral migration of such a drop, in terms of the unperturbed velocity field.

First, we solve the general velocity fields in and around a deformed drop suspended in an arbitrary (but Stokesian) unbounded flow field. The usefulness of the solution is then demonstrated by solving the drag force and lateral migration of a drop suspended in an unbounded Poiseuillian field.

2. Formulation of the problem

2.1. *Statement of the problem*

The problem considered herein is that of a single non-neutrally buoyant drop or bubble suspended in an unbounded medium. The fluids involved are isothermal, Newtonian and of constant physical properties. The flow around the drop is creeping so that inertial terms may be neglected.

The co-ordinate system employed is spherical (r, θ, ϕ), with the origin coinciding with the centre of mass of the drop. It is assumed that the radial component of the terminal settling velocity is small, so that the co-ordinate system is inertial. The co-ordinate system is depicted in figure 1.

The conservation equations exterior to the drop are

$$\mu_e \nabla^2 \mathbf{v} = \nabla p_e, \qquad (1a)$$

$$\nabla . \mathbf{v} = 0, \qquad (1b)$$

† The term drop is used here for brevity. The solution applies to bubbles as well.

and for the flow field interior to the drop

$$\mu_i \nabla^2 \mathbf{u} = \nabla p_i, \tag{2a}$$

$$\nabla . \mathbf{u} = 0, \tag{2b}$$

where $\mathbf{v}$ and $\mathbf{u}$ are the velocity vectors exterior to the drop and interior to it, respectively; p is the pressure, including the potential gravity field and the subscripts e and i refer to the properties exterior to the drop and interior to it, respectively.

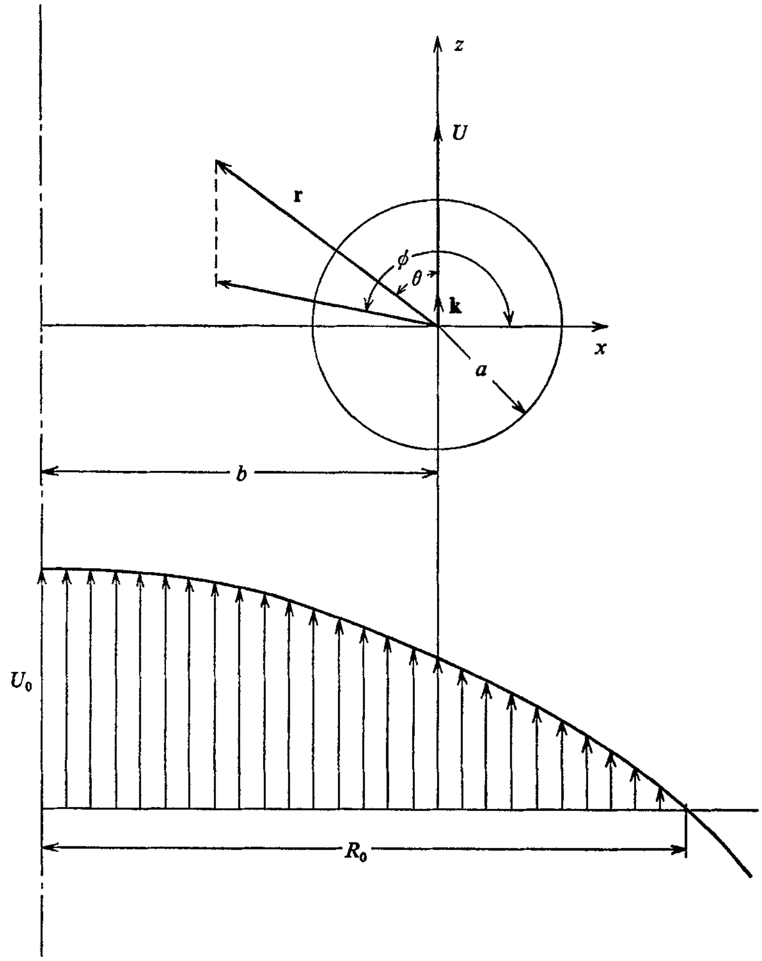

FIGURE 1. The co-ordinate system used.

2.2. *The boundary conditions*

The boundary conditions to be employed, neglecting surface-active agents, are as follows:

At the interface of the drop the following boundary conditions are specified:

$$\mathbf{v} = \mathbf{u}, \tag{3a}$$

$$v_n = u_n = 0, \tag{3b}$$

$$\boldsymbol{\pi}_{(n)} = \boldsymbol{\tau}_{(n)} + \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\mathbf{t}_n, \tag{3c}$$

where $\boldsymbol{\pi}_{(n)}$ and $\boldsymbol{\tau}_{(n)}$ are the normal vectors of the stress tensor exterior to the drop and interior to it, respectively; $\mathbf{t}_n$ is a unit vector normal to the interface; σ is the surface tension; and R_1 and R_2 are the principal radii of the interface.

The boundary conditions far from the drop are

$$\mathbf{v} = \mathbf{v}_\infty \quad \text{at} \quad r = \infty, \tag{3d}$$

where $\mathbf{v}_\infty$ is some arbitrary (but Stokesian) velocity distribution.

3. The solution

3.1. *The first iteration*

The first iteration, i.e. the flow fields in and around a *spherical* drop, was calculated in our previous work (Hetsroni & Haber 1970). Also given there is the function describing the deviation of the interface from sphericity, to a first-order approximation.

To sum up that solution briefly: the velocity and pressure fields exterior to the drop are

$$\mathbf{v}^{(1)} = \mathbf{v}_\infty + \sum_{n=1}^{\infty}\left\{\nabla\times(\mathbf{r}\chi^{(1)}_{-n-1}) + \nabla\phi^{(1)}_{-n-1} - \frac{n-2}{2n(2n-1)\mu_e}r^2\nabla p^{(1)}_{-n-1} + \frac{n+1}{n(2n-1)\mu_e}\mathbf{r}p^{(1)}_{-n-1}\right\}, \tag{4a}$$

$$p^{(1)}_e = p_\infty + \sum_{n=0}^{\infty} p^{(1)}_{-n-1}, \tag{4b}$$

where the superscript (1) denotes the first iteration. The velocity and pressure fields interior to the drop are

$$\mathbf{u}^{(1)} = \sum_{n=1}^{\infty}\left\{\nabla\times(\mathbf{r}\chi^{(1)}_{n}) + \nabla\phi^{(1)}_{n} + \frac{n+3}{2(n+1)(2n+3)\mu_i}r^2\nabla p^{(1)}_{n} - \frac{n}{(n+1)(2n+3)\mu_i}\mathbf{r}p^{(1)}_{n}\right\}, \tag{5a}$$

$$p^{(1)}_i = \sum_{n=0}^{\infty} p^{(1)}_{n}, \tag{5b}$$

where $\chi^{(1)}_n, \phi^{(1)}_n, p^{(1)}_n, \chi^{(1)}_{-n-1}, \phi^{(1)}_{-n-1}, p^{(1)}_{-n-1}$ are known solid spherical harmonics (of the first iteration). These spherical harmonics depend directly on the unperturbed velocity field $\mathbf{v}_\infty$ and on the physical properties. These solid spherical harmonics and their coefficients are defined in appendix A.

Also resulting from the first iteration is the equation of the interface of the drop, i.e.

$$r = a[1+\xi^{(1)}(\theta,\phi)], \tag{6a}$$

where $\xi^{(1)}$ is the function describing the deviation of the interface from sphericity, viz.

$$\xi^{(1)}(\theta,\phi) = \sum_{q=2}^{\infty} L_q S_q(\theta,\phi), \tag{6b}$$

where S_q are the surface harmonics, and the L_q's are small dimensionless parameters on which the second iteration is based, and which are defined in appendix A:

$$L_q^m = \frac{1}{(q^2+q-2)q(q+1)}\frac{1}{(1+\lambda)}\left\{\frac{\alpha_q^m\mu_e}{\sigma}[(4q^3+6q^2+2q+3)\lambda+(4q^3+6q^2-4q-6)]\right.$$
$$\left.+\frac{\beta_q^m\mu_e}{\sigma}[(4q^3+6q^2+2q-3)\lambda+(4q^3+6q^2-4q)]\right\}, \quad (7)$$

where $\lambda \equiv \mu_i/\mu_e$ and where α_q^m and β_q^m have dimensions of velocity and depend solely on the unperturbed velocity distribution. Notice that the above derivation was carried out for the case when both $(\alpha_q^m\mu_e/\sigma) \ll 1$ and $(\beta_q^m\mu_e/\sigma) \ll 1$. For this case our solution agrees with the presumably more general case treated by Cox (1969). This restriction is however not serious.

3.2. *Velocity fields of second iteration*

The velocity and pressure fields interior to the drop and exterior to it are now defined as follows:

$$\mathbf{u} = \mathbf{u}^{(1)}+\mathbf{u}^{(2)} = \mathbf{u}^{(1)}+\sum_{q=2}^{\infty} L_q\mathbf{u}_{(q)}, \quad (8)$$

$$p_i = p_i^{(1)}+p_i^{(2)} = p_i^{(1)}+\sum_{q=2}^{\infty} L_q p_{i(q)}, \quad (9)$$

$$\mathbf{v} = \mathbf{v}^{(1)}+\mathbf{v}^{(2)} = \mathbf{v}^{(1)}+\sum_{q=2}^{\infty} L_q\mathbf{v}_{(q)}, \quad (10)$$

$$p_e = p_e^{(1)}+p_e^{(2)} = p_e^{(1)}+\sum_{q=2}^{\infty} L_q p_{e(q)}, \quad (11)$$

where $\mathbf{u}^{(2)}$, $p_i^{(2)}$, $\mathbf{v}^{(2)}$, $p_e^{(2)}$ are the second iterations of the velocity and pressure fields interior to the drop and exterior to it, respectively. These fields were expanded in series, in terms of the small dimensionless parameters L_q, where $\mathbf{v}_{(q)}$ and $\mathbf{u}_{(q)}$ are to be determined.

The notation of equations (8) to (11) is an abbreviation of a more detailed notation, for example:

$$\mathbf{u}^{(2)} = \sum_{q=2}^{\infty} L_q\mathbf{u}_{(q)} = \sum_{q=L}^{\infty}\sum_{m=0}^{q}[L_q^m\mathbf{u}(q,m)+\hat{L}_q^m\hat{\mathbf{u}}(q,m)].$$

The equations of motion to be satisfied are obtained by substitution of (7)–(11) into (1) and (2). Equating term by term and recalling that the L_q's are independent and that $\mathbf{v}^{(1)}$, $p_e^{(1)}$, etc., satisfy (1), one finds that $\mathbf{v}_{(q)}$, $\mathbf{u}_{(q)}$ also satisfy the Stokes equations and the equation of continuity. The velocities $\mathbf{v}_{(q)}$ and $\mathbf{u}_{(q)}$ can be described by the general solution of Lamb (1945), identical to (4*a*) and (5*a*).

The boundary conditions of (3) are not convenient for computation, and are therefore transformed as follows:

$$[\mathbf{v}]_s.\mathbf{t}_n = [\mathbf{u}]_s.\mathbf{t}_n = 0, \quad (12a,b)$$

$$\nabla.[\mathbf{v}]_s = \nabla.[\mathbf{u}]_s, \quad (12c)$$

$$\mathbf{t}_n.\nabla\times[\mathbf{v}]_s = \mathbf{t}_n.\nabla\times[\mathbf{u}]_s, \quad (12d)$$

$$\mathbf{t}_n.\nabla\times[\boldsymbol{\pi}_{(n)}]_s=\mathbf{t}_n.\nabla\times[\boldsymbol{\tau}_{(n)}]_s+\mathbf{t}_n.\nabla\times\left[\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\mathbf{t}_n\right], \tag{12e}$$

$$\mathbf{t}_n.\nabla\times(\mathbf{t}_n\times[\boldsymbol{\pi}_{(n)}]_s)=\mathbf{t}_n.\nabla\times(\mathbf{t}_n\times[\boldsymbol{\tau}_{(n)}]_s)+\mathbf{t}_n.\nabla\times\left[\mathbf{t}_n\times\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\mathbf{t}_n\right], \tag{12f}$$

$$\mathbf{t}_n.[\boldsymbol{\pi}_{(n)}]_s-\mathbf{t}_n.[\boldsymbol{\tau}_{(n)}]_s=\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{12g}$$

where $\boldsymbol{\pi}_{(n)}$ and $\boldsymbol{\tau}_{(n)}$ are the normal stress vectors exterior to the drop and interior to it, respectively; σ is the surface tension; the subscript s denotes that the parenthesized value is evaluated on the interface, and where the unit vector $\mathbf{t}_n$ is, to a first-order approximation (cf. Cox 1969):

$$\mathbf{t}_n=\mathbf{t}_r-\sum_q L_q\tilde{\nabla}S_q+O(L_q^2), \tag{13}$$

where $\tilde{\nabla}(\;)$ is a dimensionless vectorial differential operator defined as

$$\tilde{\nabla}=\mathbf{t}_\theta\frac{\partial}{\partial\theta}+\mathbf{t}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}. \tag{14}$$

Substituting (8)–(11) into (12) one obtains the following boundary conditions for the second iteration of the velocity fields interior to the drop and exterior to it (appendix B):

from (B 5) $$v^*_{(q)r}=\tilde{\nabla}.(S_q\mathbf{v}^{(1)*}); \tag{15a}$$

from (B 6) $$v^*_{(q)r}-u^*_{(q)r}=0; \tag{15b}$$

from (B 9) $$\left[r\frac{\partial v_{(q)r}}{\partial r}\right]^*-\left[r\frac{\partial u_{(q)r}}{\partial r}\right]^*=\tilde{\nabla}.\left\{S_q\left[r\frac{\partial\mathbf{v}^{(1)}}{\partial r}-r\frac{\partial\mathbf{u}^{(1)}}{\partial r}\right]^*\right\}; \tag{15c}$$

from (B 13) $$\mathbf{r}.\nabla\times[\mathbf{v}_{(q)}-\mathbf{u}_{(q)}]^*=\mathbf{r}.\nabla\times\left\{S_q\left[r\frac{\partial\mathbf{u}^{(1)}}{\partial r}-r\frac{\partial\mathbf{v}^{(1)}}{\partial r}\right]^*\right\}; \tag{15d}$$

from (B 16)

$$\begin{aligned}\mathbf{t}_r.\nabla\times[\boldsymbol{\pi}_{(r)(q)}-\boldsymbol{\tau}_{(r)(q)}]^*=&\tilde{\nabla}S_q.\sum_{n=2}^{\infty}(n^2+n-2)\frac{\sigma L_n}{a}\nabla\times(S_n\mathbf{t}_r)\\&+\mathbf{t}_r.\nabla\times\{\tilde{\nabla}S_q.[\boldsymbol{\pi}^{(1)}-\boldsymbol{\tau}^{(1)}]^*\}-\mathbf{t}_r.\nabla\times\left\{S_q\left[r\frac{\partial\boldsymbol{\pi}^{(1)}_{(r)}}{\partial r}-r\frac{\partial\boldsymbol{\tau}^{(1)}_{(r)}}{\partial r}\right]^*\right\};\end{aligned} \tag{15e}$$

where $\boldsymbol{\pi}^{(1)}$ and $\boldsymbol{\tau}^{(1)}$ are the stress tensors, based on $\mathbf{v}^{(1)}$ and $\mathbf{u}^{(1)}$, respectively, from (B 19)

$$\begin{aligned}\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times[\boldsymbol{\pi}_{(r)(q)}-\boldsymbol{\tau}_{(r)(q)}]^*\}=&-\mathbf{t}_r.\nabla\times\left\{\mathbf{t}_r\times S_q\left[r\frac{\partial\boldsymbol{\pi}^{(1)}_{(r)}}{\partial r}-r\frac{\partial\boldsymbol{\tau}^{(1)}_{(r)}}{\partial r}\right]^*\right\}\\&+\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times[\boldsymbol{\pi}^{(1)}-\boldsymbol{\tau}^{(1)}]^*.\tilde{\nabla}S_q\}\\&+\mathbf{t}_r.\nabla\times\left\{\tilde{\nabla}S_q\times\left[\mathbf{t}_r(\sigma/a)\left(2+\sum_{n=2}^{\infty}(n^2+n-2)L_nS_n\right)\right]\right\}.\end{aligned} \tag{15f}$$

The solution is now continued by substituting (4*a*) and (5*a*) into the right-hand side of (15). After somewhat lengthy computation the following is obtained (appendix C):

$$\tilde{\nabla}\,.\,(S_q\mathbf{u}^{(1)*}) = \sum_{n=1}^{\infty}\left\{\sum_{i=0\text{ or }1}^{n+q} a_i^{q,n}S_i\right\}, \tag{16a}$$

$$\tilde{\nabla}\,.\left[S_q\left(r\frac{\partial\mathbf{v}^{(1)}}{\partial r}-r\frac{\partial\mathbf{u}^{(1)}}{\partial r}\right)^*\right] = \sum_{n=1}^{\infty}\{\sum_i b_i^{q,n}S_i\}, \tag{16b}$$

$$\mathbf{t}_r.\tilde{\nabla}\times\left[S_q\left(r\frac{\partial\mathbf{v}^{(1)}}{\partial r}-r\frac{\partial\mathbf{u}^{(1)}}{\partial r}\right)^*\right] = \sum_{n=1}^{\infty}\{\sum_i c_i^{q,n}S_i\}, \tag{16c}$$

$$\tilde{\nabla}S_q\,.\sum_{n=2}^{\infty}\tilde{\nabla}\times(\mathbf{t}_rS_n)(n^2+n-2)(L_n\sigma/a)+\mathbf{t}_r.\tilde{\nabla}\times[\tilde{\nabla}S_q\,.(\boldsymbol{\pi}^{(1)}-\boldsymbol{\tau}^{(1)})^*]$$
$$-\mathbf{t}_r.\tilde{\nabla}\times\left[S_q\left(r\frac{\partial\boldsymbol{\pi}_{(r)}^{(1)}}{\partial r}-r\frac{\partial\boldsymbol{\tau}_{(r)}^{(1)}}{\partial r}\right)^*\right] = \frac{\mu_e}{a}\sum_{n=1}^{\infty}\{\sum_i d_i^{q,n}S_i\}, \tag{16d}$$

$$-\mathbf{t}_r.\tilde{\nabla}\times\left[\mathbf{t}_r\times S_q\left(r\frac{\partial\boldsymbol{\pi}_{(r)}^{(1)}}{\partial r}-r\frac{\partial\boldsymbol{\tau}_{(r)}^{(1)}}{\partial r}\right)^*\right]+\mathbf{t}_r.\tilde{\nabla}\times[\mathbf{t}_r\times(\boldsymbol{\pi}^{(1)}-\boldsymbol{\tau}^{(1)})^*.\tilde{\nabla}S_q]$$
$$+\mathbf{t}_r.\tilde{\nabla}\times\left\{\tilde{\nabla}S_q\times\mathbf{t}_r\sum_{n=2}^{\infty}(\sigma L_n/a)(n^2+n-2)S_n\right\} = (\mu_e/a)\sum_{n=1}^{\infty}\{\sum_i e_i^{q,n}S_i\}, \tag{16e}$$

where the coefficients $a_i^{q,n}$, $b_i^{q,n}$, $c_i^{q,n}$, $d_i^{q,n}$ and $e_i^{q,n}$ are defined by (C 13) and (C 14) (appendix C). These coefficients are merely functions of the ratio of viscosities λ and the known coefficients of the unperturbed velocity field α_n, β_n and γ_n. Note that in (15) we expressed the unknown velocity fields and combinations thereof (on the left-hand sides) in terms of known quantities, on the right-hand sides of the equations. In order to simplify further this presentation, new coefficients $\tilde{a}_n^q$, $\tilde{b}_n^q$, $\tilde{c}_n^q$, $\tilde{d}_n^q$ and $\tilde{e}_n^q$ are now defined as follows:

$$\tilde{a}_{n(q)} = \left\{\begin{array}{ll}\sum_{i=1}^{\infty} a_n^{q,i} & \text{for}\quad n<q,\\ \sum_{i=1}^{\infty} a_n^{q,i+n-q} & \text{for}\quad n\geqslant q,\end{array}\right\} \tag{17}$$

but since

$$\sum_{n=1}^{\infty}\left\{\sum_{i=0,1}^{n+q}a_i^{q,n}S_i\right\} = \sum_{n=0,1}^{q-1}\left\{\sum_{i=1}^{\infty}a_n^{q,i}\right\}S_n+\sum_{n=q}^{\infty}\left\{\sum_{i=1}^{\infty}a_n^{q,i+n-q}\right\}S_n,$$

one obtains
$$\sum_{n=1}^{\infty}\left\{\sum_{i=0,1}^{n+q}a_i^{q,n}S_i\right\} = \sum_{n=1}^{\infty}\tilde{a}_{n(q)}S_n. \tag{18}$$

The coefficients $\tilde{b}_{n(q)}$, $\tilde{c}_{n(q)}$, $\tilde{d}_{n(q)}$ and $\tilde{e}_{n(q)}$ are defined in a similar manner. Thus, substituting (4), (8)–(11) and (18) in (15), the following is obtained (see also Hetsroni & Haber 1970):

$$\sum_{n=1}^{\infty}\left\{\frac{na}{2(2n+3)\mu_i}p_{n(q)}^*+\frac{n}{a}\Phi_{n(q)}^*\right\} = \sum_{n=1}^{\infty}\tilde{a}_{n(q)}S_n, \tag{19a}$$

$$\sum_{n=1}^{\infty}\left\{\frac{na}{2(2n+3)\mu_i}p_{n(q)}^*+\frac{n}{a}\Phi_{n(q)}^*-\frac{(n+1)a}{2(2n-1)\mu_e}p_{-n-1(q)}^*+\frac{n+1}{a}\Phi_{-n-1(q)}^*\right\} = 0, \tag{19b}$$

$$\sum_{n=1}^{\infty}\left\{-\frac{n(n+1)a}{2(2n-1)\mu_e}p^*_{-n-1(q)}+\frac{(n+1)(n+2)}{a}\Phi^*_{-n-1(q)}-\frac{n(n+1)a}{2(2n+3)\mu_i}p^*_{n(q)}\right.$$
$$\left.-\frac{n(n-1)}{a}\Phi^*_{n(q)}\right\}=\sum_{n=1}^{\infty}\tilde{b}_{n(q)}S_n, \quad (19c)$$

$$\sum_{n=1}^{\infty} n(n+1)[\chi^*_{-n-1(q)}-\chi^*_{n(q)}]=\sum_{n=1}^{\infty}\tilde{c}_{n(q)}S_n, \quad (19d)$$

$$\sum_{n=1}^{\infty} n(n+1)[-\lambda(n-1)\chi^*_{n(q)}-(n+2)\chi^*_{-n-1(q)}]=\sum_{n=1}^{\infty}\tilde{d}_{n(q)}S_n, \quad (19e)$$

$$\sum_{n=1}^{\infty}\left\{\frac{2n(n+1)(n+2)}{a}\Phi^*_{-n-1(q)}-\frac{(n+1)^2(n-1)a}{(2n-1)\mu_e}p^*_{-n-1(q)}\right.$$
$$\left.+\frac{2\lambda}{a}(n-1)(n+1)n\Phi^*_{n(q)}+\frac{n^2(n+2)a\lambda}{(2n+3)\mu_i}p^*_{n(q)}\right\}=\sum_{n=1}^{\infty}\tilde{e}_{n(q)}S_n. \quad (19f)$$

The unknown solid spherical harmonics $p_{n(q)}$, $\Phi_{n(q)}$, $\chi_{n(q)}$, $p_{-n-1(q)}$, $\Phi_{-n-1(q)}$ and $\chi_{-n-1(q)}$ are defined as follows:

$$\left.\begin{aligned}
p_{n(q)}&=A_{n(q)}\mu_i a^{-n-1}r^n S_n(\theta,\phi); & p_{-n-1(q)}&=A_{-n-1(q)}\mu_e a^n r^{-n-1}S_n(\theta,\phi);\\
\Phi_{n(q)}&=B_{n(q)}a^{-n+1}r^n S_n(\theta,\phi); & \Phi_{-n-1(q)}&=B_{-n-1(q)}a^{n+2}r^{-n-1}S_n(\theta,\phi);\\
\chi_{n(q)}&=C_{n(q)}a^{-n}r^n S_n(\theta,\phi); & \chi_{-n-1(q)}&=C_{-n-1(q)}a^{n+1}r^{-n-1}S_n(\theta,\phi).
\end{aligned}\right\} \quad (20)$$

Again, one has to keep in mind that, in this presentation, $A_{n(q)}S_n(\theta,\phi)$ is actually an abbreviation for $2n+1$ terms, that is,

$$A_{n(q)}S_n(\theta,\phi)=\sum_{m=0}^{n}[A^m_{n(q)}\cos(m\phi)+\hat{A}^m_{n(q)}\sin(m\phi)]P^m_n(\cos\theta). \quad (21)$$

The coefficients

$$A^m_{n(q)},\quad B^m_{n(q)},\quad C^m_{n(q)},\quad A^m_{-n-1(q)},\quad B^m_{-n-1(q)},\quad C^m_{-n-1(q)},\ \text{etc.}$$

are unknown and have to be determined. Substituting (20) into (19) there are obtained six equations with the six unknowns. These equations can be separated into two groups of equations, viz.

$$\begin{bmatrix}
\dfrac{1}{2(2n+3)} & 1 & 0 & 0\\
0 & 0 & \dfrac{1}{2(2n-1)} & -1\\
\dfrac{n(n+1)}{2(2n+3)} & n(n-1) & \dfrac{n(n+1)}{2(2n-1)} & -(n+1)(n+2)\\
\dfrac{n^2(n+2)\lambda}{2n+3} & 2\lambda(n-1)n(n+1) & -\dfrac{(n+1)^2(n-1)}{(2n-1)} & 2n(n+1)(n+2)
\end{bmatrix}
\begin{bmatrix} A_{n(q)}\\ B_{n(q)}\\ A_{-n-1(q)}\\ B_{-n-1(q)}\end{bmatrix}
=\begin{bmatrix}\dfrac{\tilde{a}_{n(q)}}{n}\\ \dfrac{\tilde{a}_{n(q)}}{n+1}\\ -\tilde{b}_{n(q)}\\ \tilde{e}_{n(q)}\end{bmatrix} \quad (22a)$$

and

$$\begin{bmatrix}-1 & 1\\ \lambda(n-1) & (n+2)\end{bmatrix}\begin{bmatrix}C_{n(q)}\\ C_{-n-1(q)}\end{bmatrix}=\begin{bmatrix}\dfrac{\tilde{c}_{n(q)}}{n(n+1)}\\ -\dfrac{\tilde{d}_{n(q)}}{n(n+1)}\end{bmatrix}. \quad (22b)$$

The solution of the above equations results in

$$\left.\begin{aligned}
A_{n(q)} &= \frac{2n+3}{n(2n+1)(1+\lambda)}[\tilde{e}_{n(q)}-(2n^2+1)\tilde{a}_{n(q)}-(2n+1)\tilde{b}_{n(q)} \\
&\qquad -2\lambda(n^2-1)\tilde{a}_{n(q)}], \\
B_{n(q)} &= \frac{-1}{2n(2n+1)(1+\lambda)}[\tilde{e}_{n(q)}-(2n^2+4n+3)\tilde{a}_{n(q)}-(2n+1)\tilde{b}_{n(q)} \\
&\qquad -2\lambda n(n+2)\tilde{a}_{n\,q)}], \\
A_{-n-1(q)} &= \frac{2n-1}{(2n+1)(n+1)(1+\lambda)}\{\tilde{e}_{n(q)}+2n(n+2)\tilde{a}_{n(q)} \\
&\qquad +\lambda[(2n+1)\tilde{b}_{n(q)}+(2n^2+4n+3)\tilde{a}_{n(q)}]\}, \\
B_{-n-1(q)} &= \frac{1}{2(2n+1)(n+1)(1+\lambda)}\{\tilde{e}_{n(q)}+2(n^2-1)\tilde{a}_{n(q)} \\
&\qquad +\lambda[(2n+1)\tilde{b}_{n(q)}+(2n^2+1)\tilde{a}_{n(q)}]\}, \\
C_{n(q)} &= -\frac{\tilde{d}_{n(q)}+(n+2)\tilde{c}_{n(q)}}{n(n+1)[\lambda(n-1)+(n+2)]}, \\
C_{-n-1(q)} &= -\frac{\tilde{d}_{n(q)}-\lambda(n-1)\tilde{c}_{n(q)}}{n(n+1)[\lambda(n-1)+(n+2)]}.
\end{aligned}\right\} \quad (23)$$

Thus, the solutions of the second iteration of the velocity fields are completed, with (4), (8)–(11), (20) and (23). These solutions are given in terms of the known coefficients of the unperturbed velocity field far from the drop.

Note that the solution thus far has been kept quite general: it gives the velocity fields inside and around a single *deformed* drop suspended in an unbounded medium when the unperturbed velocity field far from the drop is arbitrary (albeit Stokesian).

The general solution is now continued and the general drag force on a deformed drop is evaluated. An example of the use of our solution for an unbounded parabolic flow is given in §3.4.

3.3. *The drag force on a deformed drop*

The drag force acting on a deformed drop is now evaluated. We avail ourselves of the formula of Happel & Brenner (1965, p. 67) expressing the drag force in terms of the solid spherical harmonic p_{-2} and recall that this drag is independent of the shape of the drop:

$$\mathbf{F}_D = -4\pi\nabla(r^3 p_{-2}).$$

For the flow field of the second iteration, namely the force acting on the drop due to the velocity $\mathbf{v}^{(2)}$ of the second iteration only, one obtains:

$$\mathbf{F}_D^{(2)} = -4\pi \sum_{q=2}^{\infty} L_q \nabla(r^3 p_{-2(q)}). \quad (24)$$

Substituting (20), (21) and (23) into (24) one obtains

$$\mathbf{F}_D^{(2)} = -4\pi\mu_e a \sum_{q=2}^{\infty} L_q\{A^0_{-2(q)}\mathbf{k}+\hat{A}^1_{-2(q)}\mathbf{j}+A^1_{-2(q)}\mathbf{i}\}, \quad (25)$$

where $$A^m_{-2(q)} = \frac{1}{6(1+\lambda)}\{\tilde{e}^m_{1(q)}+3\lambda\tilde{b}^m_{1(q)}+3(2+3\lambda)\tilde{a}^m_{1(q)}\} \quad (m=0,1,\hat{1}), \quad (26)$$

which completes the formula for the general drag force acting on a single deformed drop due to the velocity $\mathbf{v}^{(2)}$ only. Again, the force $\mathbf{F}_D^{(2)}$ is given in terms of the coefficients $\tilde{a}_{1(q)}^m$ and $\tilde{e}_{1(q)}^m$ which in turn depend on the physical properties of the fluids and on the unperturbed velocity distribution far from the drop.

3.4. *A drop suspended in an unbounded Poiseuillian flow field*

To illustrate the usefulness of our general solution we shall now consider the forces acting on a drop suspended in an unbounded constant Poiseuillian velocity field.

This problem is solved as follows. First, the coefficients α_n, β_n and γ_n of the unperturbed velocity field are computed (appendix A); next, the coefficients

	$i \diagdown l$	1	2	3
${}^{l}\aleph_i$	1	$\frac{5}{2(1+\lambda)}$	$\frac{15}{2}\frac{1-\lambda}{1+\lambda}$	$\frac{5}{2}\frac{2-3\lambda}{1+\lambda}$
${}^{l}\beth_i$	1	$\frac{1}{2(1+\lambda)}$	$\frac{3}{2}\frac{1-\lambda}{1+\lambda}$	$\frac{-15\lambda}{2(1+\lambda)}$
	2	$\frac{1}{2(1+\lambda)}$	$\frac{5}{2}\frac{1-\lambda}{1+\lambda}$	$\frac{6-41\lambda}{2(1+\lambda)}$
	3	$\frac{5}{12}\frac{1}{1+\lambda}$	$\frac{35}{12}\frac{1-\lambda}{1+\lambda}$	$\frac{5}{12}\frac{16-79\lambda}{1+\lambda}$
${}^{l}\gimel_i$	1	$\frac{1}{2}$	0	0

TABLE 1

designated by the Hebrew letters, Aleph ${}^{l}\aleph_n$, Beth ${}^{l}\beth_n$ and Gimel ${}^{l}\gimel_n$ $(l = 1,2,3)$ are calculated by using (C 2), (C 5) and (C 9); finally, the corresponding coefficients $h_1^{q,n}$, $g_1^{q,n}$, $f_1^{q,n}$, $n_1^{q,n}$ and $m_1^{q,n}$ are evaluated from (C 14).

The coefficients of the first iteration, for the deviation from sphericity function, are given in appendix A as

$$L_2^1 = \frac{16+19\lambda}{8(1+\lambda)}\frac{\mu_e}{\sigma}\beta_2^1, \quad L_3^0 = \frac{10+11\lambda}{8(1+\lambda)}\frac{\mu_e}{\sigma}\beta_3^0. \tag{27}$$

These coefficients were evaluated by Hetsroni & Haber (1970) for a Poiseuillian flow as follows:

$$\left.\begin{aligned} \alpha_1^0 &= -\tfrac{2}{5}U_0\left(\frac{a}{R_0}\right)^2, \quad \beta_1^0 = -U_{\mathrm{HR}} + \frac{2\lambda}{2+3\lambda}U_0\left(\frac{a}{R_0}\right)^2, \\ \beta_2^1 &= -\tfrac{2}{3}U_0\frac{ab}{R_0^2}, \quad \beta_3^0 = \tfrac{2}{5}U_0\left(\frac{a}{R_0}\right)^2, \quad \hat{\gamma}_1^1 = U_0\frac{ab}{R_0^2}, \end{aligned}\right\} \tag{28}$$

where the terminal settling velocity of Hadamard–Rybczynski is defined in the usual way, namely

$$\mathrm{U}_{\mathrm{HR}} = \frac{(\rho_i-\rho_e)\mathrm{g}a^2}{\mu_e}\frac{2(1+\lambda)}{3(2+3\lambda)}.$$

The corresponding coefficients ${}^{l}\aleph_n$, ${}^{l}\beth_n$ and ${}^{l}\gimel_n$ can now be evaluated. This was done, and the results are tabulated in table 1.

The corresponding coefficients $h_1^{q,i}$, $g_1^{q,i}$, $f_1^{q,i}$, $l_1^{q,i}$ and $m_1^{q,i}$ are calculated according to (C 14) and are summarized in table 2, where X denotes a coefficient which is not relevant to the following calculation. Note that non-zero results for $h_1^{q,i}$, $g_1^{q,i}$ and $e_1^{q,i}$ were obtained as coefficients of the surface harmonic $\cos\phi\, P_1^1(\cos\theta)$ only.

Substituting (27) and (28) with the values from tables 1 and 2 into (C 13), one obtains $\tilde{e}^1_{1(q)}$ and $\tilde{a}^1_{1(q)}$ and $\tilde{b}^1_{1(q)}$ for $q = 2, 3$. Using (25) and (26) the drag force of the second iteration is obtained:

$$\mathbf{F}_D^{(2)} = \mathbf{i}\,\frac{\pi\mu_e a}{420(1+\lambda)^3}\frac{\mu_e U_0}{\sigma}\frac{a}{R_0}\frac{b}{R_0}[21(19\lambda+16)(3\lambda^2+3\lambda+4)\beta_1^0 + (2985\lambda^3+1971\lambda^2+2194\lambda+2440)\alpha_1^0], \qquad (29)$$

where α_1^0 and β_1^0 are defined by (28).

	$h_1^{q,i}$		$g_1^{q,i}$		$f_1^{q,i}$		$e_1^{q,i}$		$m_1^{q,i}$	
$i \diagdown q$ ...	2	3	2	3	2	3	2	3	2	3
1	$\frac{9}{5}$	0	$\frac{3}{5}$	0	0	0	$-\frac{9}{5}$	0	0	0
2	0	$-\frac{72}{35}$	0	$-\frac{9}{35}$	X	X	0	0	X	X
3	$-\frac{72}{35}$	0	$-\frac{9}{35}$	0	X	X	$\frac{432}{35}$	0	X	X

TABLE 2

It is important to realize that the second iteration for Poiseuillian flow resulted in a drag force in the *radial direction only.*

In order to evaluate the radial migration of the droplet (the radial component of the terminal settling velocity), we recall the functional relationship between the radial velocity U_x and the resulting drag force:

$$F_{Dx} = -2\pi\mu_e a U_x(2+3\lambda)/(1+\lambda). \qquad (30)$$

Equating (29) to (30), the radial velocity U_x is obtained as follows:

$$U_x = -\frac{1}{840(1+\lambda)^2(2+3\lambda)}(\text{מ})\frac{ab}{R_0^2}[21(19\lambda+16)(3\lambda^2+3\lambda+4)\beta_1^0 + (2985\lambda^3+1971\lambda^2+2194\lambda+2440)\alpha_1^0] + O(\text{מ}^2), \qquad (31)$$

where the Hebrew letter Mem defines a small dimensionless group as follows:

$$(\text{מ}) = \mu_e U_0/\sigma.$$

Substituting (28) into (31) one obtains the radial component of the terminal settling velocity of a deformed drop as follows:

$$U_x = \frac{(19\lambda+16)(3\lambda^2+3\lambda+4)}{40(2+3\lambda)(1+\lambda)^2}(\text{מ})\,U_{\mathrm{HR}}\frac{a}{R_0}\frac{b}{R_0} + \frac{(1485\lambda^4+429\lambda^3-1248\lambda^2+2494\lambda+2440)}{1050(2+3\lambda)^2(1+\lambda)^2}(\text{מ})\,U_0\left(\frac{a}{R_0}\right)^3\frac{b}{R_0} + O(\text{מ}^2),$$

or

$$U_x = K U_0 \frac{b}{R_0} + O(\text{מ}^2), \qquad (32)$$

where
$$K \equiv \frac{(19\lambda+16)(3\lambda^2+3\lambda+4)}{40(2+3\lambda)(1+\lambda)^2}(\text{מ})\frac{U_{\mathrm{HR}}}{U_0}\frac{a}{R_0} + \frac{(1485\lambda^4+429\lambda^3-1248\lambda^2+2494\lambda+2440)}{1050(2+3\lambda)^2(1+\lambda)^2}(\text{מ})\left(\frac{a}{R_0}\right)^3. \tag{33}$$

Equation (32) completes the solution for the radial component of the terminal settling velocity of a drop suspended in an unbounded Poiseuillian velocity field.

It is of interest to compute the trajectory of a drop (or bubble), when it is initially placed at a plane $Z = 0$ at a distance $\beta_0 = b_0/R_0$ from the point of maximum velocity. This can be done by solving the relation

$$Z/R_0 = \int_{\beta_0}^{\beta} (U_z/U_x)\,d\beta, \tag{34}$$

with
$$\beta = \beta_0 \quad \text{at} \quad Z/R_0 = 0,$$

where
$$\beta = b/R_0$$

and where U_z is the axial component of the terminal settling velocity. Since the second iteration does not contribute any velocity in the z direction, U_z is also the terminal settling velocity of the first iteration as determined by Hetsroni & Haber (1970), viz.

$$U_z = U_{\mathrm{HR}} + U_0\left[1-\left(\frac{b}{R_0}\right)^2 - \frac{2\lambda}{2+3\lambda}\left(\frac{a}{R_0}\right)^2\right]. \tag{35}$$

Substituting (32) and (35) into (34) we obtain

$$\frac{Z}{R_0}(K) = \int_{\beta_0}^{\beta}\frac{1}{\beta}\left[\frac{U_{\mathrm{HR}}}{U_0}+1-\beta^2-\frac{2\lambda}{2+3\lambda}\left(\frac{a}{R_0}\right)^2\right]d\beta, \tag{36}$$

where (K) is defined by (33). Solving and simplifying,

$$\frac{Z}{R_0}(K) = \left[1+\frac{U_{\mathrm{HR}}}{U_0}-\frac{2\lambda}{2+3\lambda}\left(\frac{a}{R_0}\right)^2\right]\ln\left(\frac{\beta}{\beta_0}\right)-\frac{\beta^2-\beta_0^2}{2}. \tag{37}$$

This completes the solution of the lateral migration of a drop initially placed at a distance β_0 from the point of maximum velocity of an unbounded Poiseuillian velocity field. The trajectories of drops for various cases are depicted in figure 2, where $\beta_0 = 0{\cdot}5$, and where the expression $[2\lambda(a/R_0)^2/(2+3\lambda)]\ln(\beta/\beta_0)$ was neglected in comparison to the other terms. The trajectories were computed for eight ratios of U_{HR}/U_0.

It can be noted that the directions of the radial migration depend primarily on the sign of the ratio (or product) of U_0 (the maximum unperturbed velocity) and U_{HR} (the Hadamard–Rybczynski terminal settling velocity), since the second term in (33) is small. The drop migrates away from the centreline (i.e. from $b = 0$) when this ratio is positive, and towards the centreline when the ratio is negative.

In this work we took the positive z direction to point upwards. Thus, if a bubble is placed at $\beta_0 = 0{\cdot}5$ in a Poiseuillian flow field with positive U_0, U_{HR} is also positive and the bubble will migrate away from the centreline of the Poiseuillian velocity distribution, if $\rho_i < \rho_e$.

Also shown in figure 2 is the trajectory of a neutrally buoyant drop (i.e. $U_{HR} = 0$). For this particular case we have

$$U_x = K_1 U_0 \beta,$$

where $$K_1 = \frac{(1485\lambda^4 + 429\lambda^3 - 1248\lambda^2 + 2494\lambda + 2440)}{1050(2+3\lambda)^2(1+\lambda)^2} (\text{מ}) \left(\frac{a}{R_0}\right)^3.$$

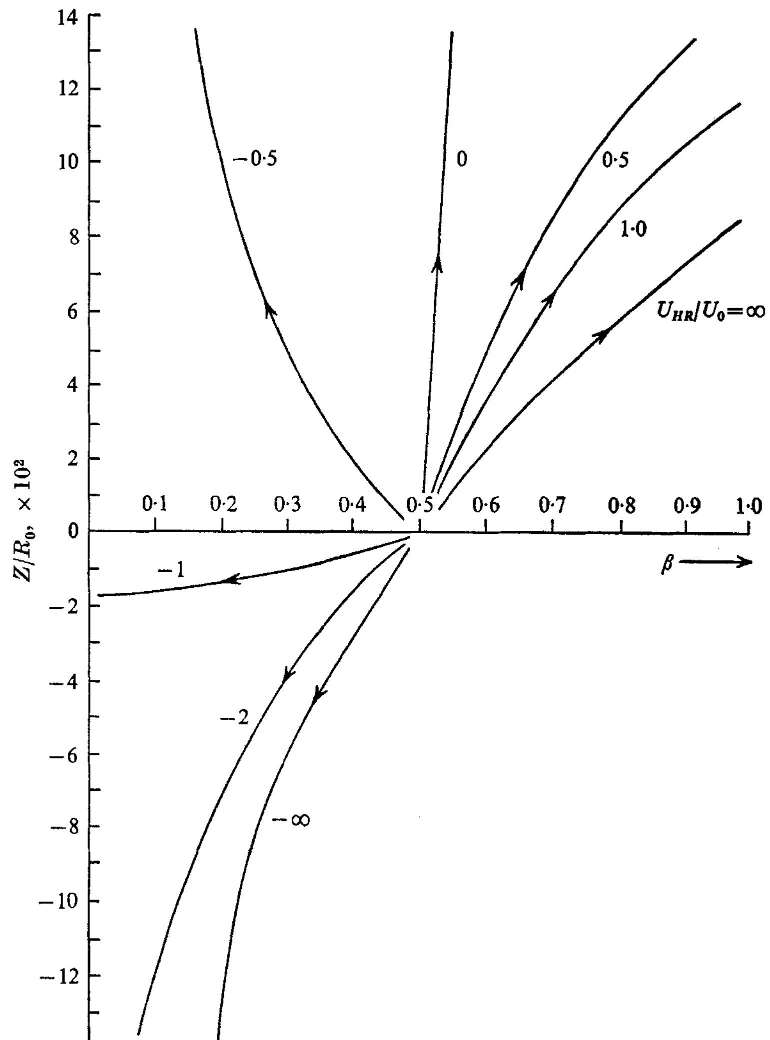

FIGURE 2. Trajectories of a drop initially located at $\beta_0 = 0{\cdot}5$ for various U_{HR}/U_0. $\mu_e U_0/\sigma = 0{\cdot}01$, $a/R_0 = 0{\cdot}1$, $\lambda = 0$.

The trajectory is determined as before, viz.

$$\frac{Z}{R_0}(K_1) = \left[1 - \frac{2\lambda}{2+3\lambda}\left(\frac{a}{R_0}\right)^2\right] \ln\left(\frac{\beta}{\beta_0}\right) - \frac{\beta^2 - \beta_0^2}{2}.$$

Note that in this case the radial component of the terminal settling velocity always takes a positive sign.

4. Conclusions

A general second-order theory is presented for deformable particles suspended in an unbounded medium when the flow field far from the particle is arbitrary (albeit Stokesian).

As a special example, the motion of a deformable particle suspended in an unbounded Poiseuillian flow field was solved. The following can be concluded from this solution.

Due to the deformation of the drop, there exists a radial component of the settling velocity. The direction of radial migration depends primarily on the sign of the product U_{HR} (the Hadamard–Rybczynski terminal settling velocity) by U_0 (the maximum Poiseuillian velocity). A positive product results in a lateral migration away from the location of maximum velocity, while the inverse is true for a negative product.

The direction of migration of neutrally buoyant particles is always away from the centreline.

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Appendix A. Recapitulation of the first iteration

The solid spherical harmonics of the first iteration (spherical droplet) were solved by Hetsroni & Haber (1970) in terms of the following harmonics:

$$\left.\begin{aligned} p_n^{(1)} &= \mu_i A_n a^{-n-1} r^n S_n(\theta,\phi); & p_{-n-1}^{(1)} &= \mu_e A_{-n-1} a^n r^{-n-1} S_n(\theta,\phi);\\ \Phi_n^{(1)} &= B_n a^{-n+1} r^n S_n(\theta,\phi); & \Phi_{-n-1}^{(1)} &= B_{-n-1} a^{n+2} r^{-n-1} S_n(\theta,\phi);\\ \chi_n^{(1)} &= C_n a^{-n} r^n S_n(\theta,\phi); & \chi_{-n-1}^{(1)} &= C_{-n-1} a^{n+1} r^{-n-1} S_n(\theta,\phi); \end{aligned}\right\} \quad \text{(A 1)}$$

where $n = 1$ to ∞ and where the coefficients were defined as follows:

$$\left.\begin{aligned} A_n^m &= \frac{(2n-1)(2n+3)}{n(1+\lambda)}\left[\frac{2n+3}{2n-1}\alpha_n^m + \beta_n^m\right],\\ B_n^m &= -\frac{A_n^m}{2(2n+3)},\\ C_n^m &= \gamma_n^m \frac{2n+1}{n(n+1)[\lambda(n-1)+n+2]},\\ A_{-n-1}^m &= -\frac{2n-1}{(n+1)(1+\lambda)}\{2\beta_n^m + \lambda[\alpha_n^m(2n+3)+\beta_n^m(2n+1)]\},\\ B_{-n-1}^m &= \frac{1}{2(n+1)(1+\lambda)}\{2\alpha_n^m - \lambda[\alpha_n^m(2n+1)+\beta_n^m(2n-1)]\},\\ C_{-n-1}^m &= \frac{\gamma_n^m(n-1)(1-\lambda)}{n(n+1)[\lambda(n-1)+n+2]}. \end{aligned}\right\} \quad \text{(A 2)}$$

The coefficients α_n^m, β_n^m and γ_n^m depend solely on the unperturbed flow field and on the geometry of the drop. These coefficients can be computed by relatively simple mathematical operations.

The equation of the interface resulting from the first iteration was given as follows:

$$r = a\left[1+\sum_{q=2}^{\infty} L_q S_q(\theta,\phi)\right], \tag{A 3}$$

where one must keep in mind that $L_q S_q(\theta,\phi)$ is an abbreviation for $2n+1$ terms, that is:

$$L_q S_q(\theta,\phi) = \sum_{m=0}^{q} [L_q^m \cos m\phi + \hat{L}_q^m \sin m\phi]\, P_q^m(\cos\theta), \tag{A 4}$$

where $P_q^m(\cos\theta)$ is the associated Legendre polynomial. The coefficients L_q^m were determined by us previously as follows:

$$L_q^m = \frac{1}{(q^2+q-2)\,q(q+1)}\frac{1}{1+\lambda}\left\{\frac{\alpha_q^m\mu_e}{\sigma}[(4q^3+6q^2+2q+3)\,\lambda+(4q^3+6q^2-4q-6)]\right.$$
$$\left.+\frac{\beta_q^m\mu_e}{\sigma}[(4q^3+6q^2+2q-3)\,\lambda+(4q^3+6q^2-4q)]\right\}, \tag{A 5}$$

with similar expressions for $\hat{L}_q^m$, when $\hat{\alpha}_q^m$ and $\hat{\beta}_q^m$ replace α_q^m and β_q^m, respectively.

Appendix B. Boundary conditions of deformed drops

Using the transformed boundary conditions (12), with (8) and (10), one can obtain the boundary conditions of a deformed drop in a convenient form.

First we use a Taylor series expansion to perform a preliminary calculation which will prove to be useful later. Expand an arbitrary vector $\mathbf{A}$, to be evaluated at the surface of the drop s in terms of its value on the sphere $r = a$ (indicated by an asterisk):

$$[\mathbf{A}]_s = [\mathbf{A}]^* + [\partial\mathbf{A}/\partial r]^* a \sum_q L_q S_q + O(L_q^2). \tag{B 1}$$

From $(12a, b)$ we have

$$[\mathbf{u}]_s\,.\,\mathbf{t}_n = 0.$$

Using (8) and (13) one obtains

$$[\mathbf{u}^{(1)}]_s\,.\,[\mathbf{t}_r - \Sigma L_q\tilde{\nabla} S_q] + \Sigma L_q[\mathbf{u}_{(q)}]_s\,.\,[\mathbf{t}_r - \Sigma L_q\tilde{\nabla} S_q] = 0, \tag{B 2}$$

which, together with the expansion demonstrated in (B 1), yields

$$\mathbf{u}^{(1)*}\,.\,\mathbf{t}_r + \mathbf{t}_r\,.\,(\partial\mathbf{u}^{(1)}/\partial r)^* a \sum_q L_q S_q - \mathbf{u}^{(1)*}\,.\sum_q L_q\tilde{\nabla} S_q + \sum_q L_q u^*_{r(q)} + O(L_q^2) = 0, \tag{B 3}$$

but, since $\mathbf{u}^{(1)*}\,.\,\mathbf{t}_r = 0$ and since the coefficients L_q are independent, the following results:

$$u^*_{(q)r} = \mathbf{u}^{(1)*}\,.\,\tilde{\nabla} S_q - S_q[r\,\partial u_r^{(1)}/\partial r]^* = \tilde{\nabla}\,.\,(S_q\mathbf{u}^{(1)*}). \tag{B 4}$$

Using an analogous procedure for the radial component of the velocity exterior to the drop, we obtain

$$v^*_{(q)r} = \mathbf{v}^{(1)*}\,.\,\tilde{\nabla} S_q - S_q[r\,\partial v_r^{(1)*}/\partial r] = \tilde{\nabla}\,.\,(S_q\mathbf{v}^{(1)*}), \tag{B 5}$$

but since $$\mathbf{v}^{(1)*} = \mathbf{u}^{(1)*},$$

we readily obtain $$v^*_{(q)r} - u^*_{(q)r} = 0. \tag{B 6}$$

In (12*c*) the velocity gradients were equated,

$$\nabla . [\mathbf{v}]_s = \nabla . [\mathbf{u}]_s. \tag{B 7}$$

Expanding $[\mathbf{v}]_s$ in a series similar to (B 1), and recalling (8) and (10), we have

$$\nabla . [\mathbf{v}]_s = \nabla . [\mathbf{v}^{(1)*}] + \nabla . [(r\,\partial \mathbf{v}^{(1)}/\partial r)^* \sum_q L_q S_q] + \nabla . (\sum_q L_q \mathbf{v}^*_{(q)}) + O(L_q^2), \tag{B 8}$$

and similarly for $\nabla . [\mathbf{u}]_s$. Substitution in (B 7), recalling that $\mathbf{v}^{(1)*} = \mathbf{u}^{(1)*}$, results in

$$\left[r\frac{\partial v_{(q)r}}{\partial r}\right]^* - \left[r\frac{\partial u_{(q)r}}{\partial r}\right]^* = \tilde{\nabla} . \left\{S_q\left[r\frac{\partial \mathbf{v}^{(1)}}{\partial r} - r\frac{\partial \mathbf{u}^{(1)}}{\partial r}\right]^*\right\}, \tag{B 9}$$

where use have been made of the equation

$$-r\nabla . \mathbf{v}^*_{(q)} = [r\,\partial v_{(q)r}/\partial r]^*.$$

The next transformed boundary condition, from (12*d*), is

$$\mathbf{t}_n . \nabla \times [\mathbf{v}]_s = \mathbf{t}_n . \nabla \times [\mathbf{u}]_s.$$

Substitution of (13) yields

$$(\mathbf{t}_r - \sum_q L_q \tilde{\nabla} S_q) . \nabla \times [\mathbf{v}]_s = (\mathbf{t}_r - \sum_q L_q \tilde{\nabla} S_q) . \nabla \times [\mathbf{u}]_s,$$

from which one obtains

$$\mathbf{r} . \nabla \times [\mathbf{v}]_s = \mathbf{r} . \nabla \times [\mathbf{u}]_s. \tag{B 10}$$

Expanding in series and using (8) and (10):

$$\begin{aligned} \mathbf{r} . \nabla \times [\mathbf{v}]_s &= \mathbf{r} . \nabla \times \{[\mathbf{v}^{(1)}]_s + \sum_q L_q \mathbf{v}^*_{(q)}\} \\ &= \mathbf{r} . \nabla \times \mathbf{v}^{(1)*} + \mathbf{r} . \nabla \times [(r\,\partial \mathbf{v}^{(1)}/\partial r)^* \sum_q L_q S_q] + \mathbf{r} . \nabla \times (\sum_q L_q \mathbf{v}^*_{(q)}), \end{aligned} \tag{B 11}$$

and similarly for $\mathbf{u}$, viz.

$$\mathbf{r} . \nabla \times [\mathbf{u}]_s = \mathbf{r} . \nabla \times \mathbf{u}^{(1)*} + \mathbf{r} . \nabla \times [(r\,\partial \mathbf{u}^{(1)}/\partial r)^* \sum_q L_q S_q] + \mathbf{r} . \nabla \times (\sum_q L_q \mathbf{u}^*_{(q)}). \tag{B 12}$$

Substitution of (B 11) and (B 12) into (B 10), recalling again that $\mathbf{v}^{(1)*} = \mathbf{u}^{(1)*}$ and that the L_q's are independent, yields

$$\mathbf{r} . \nabla \times [\mathbf{v}_{(q)} - \mathbf{u}_{(q)}]^* = \mathbf{r} . \nabla \times \left\{S_q\left[r\frac{\partial \mathbf{u}^{(1)}}{\partial r} - r\frac{\partial \mathbf{v}^{(1)}}{\partial r}\right]^*\right\}. \tag{B 13}$$

The next transformed boundary condition (12*e*) is

$$\mathbf{t}_n . \nabla \times [\boldsymbol{\pi}_{(n)}]_s = \mathbf{t}_n . \nabla \times [\boldsymbol{\tau}_{(n)}]_s + \mathbf{t}_n . \nabla \times \left[\sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\mathbf{t}_n\right],$$

from which one has

$$\mathbf{t}_n . \nabla \times [\boldsymbol{\pi}_{(n)}]_s = \mathbf{t}_n . \nabla \times [\boldsymbol{\tau}_{(n)}]_s. \tag{B 14}$$

Recalling the relations

$$\begin{aligned} [\boldsymbol{\pi}_{(n)}]_s = [\pi]_s . \mathbf{t}_n &= [\pi]^* . (\mathbf{t}_r - \sum_q L_q \tilde{\nabla} S_q) + [\sum_q L_q S_q (r\,\partial \pi/\partial r)^*] . (\mathbf{t}_r - \sum_q L_q \tilde{\nabla} S_q) \\ &= \boldsymbol{\pi}^*_{(r)} - \sum_q L_q \pi^* . \tilde{\nabla} S_q + \sum_q L_q S_q (r\,\partial \boldsymbol{\pi}_{(r)}/\partial r)^* + O(L_q^2), \end{aligned}$$

where the stress tensor π can be expressed in a manner similar to (10), i.e.

$$\pi = \pi^{(1)} + \Sigma L_q \pi_{(q)}.$$

Thus,

$$[\boldsymbol{\pi}_{(n)}]_s = \boldsymbol{\pi}^{(1)*}_{(r)} + \sum_q L_q \boldsymbol{\pi}_{(r)(q)} - \sum_q L_q \pi^{(1)}.\tilde{\nabla} S_q + \sum_q L_q S_q (r\,\partial \boldsymbol{\pi}^{(1)}_{(r)}/\partial r)^* + O(L_q^2). \quad \text{(B 15)}$$

The normal stress vector inside the droplet $\boldsymbol{\tau}_{(n)}$ evaluated on the surface of the drop is determined analogously. Substituting the above in the boundary condition (B 14), one obtains

$$\begin{aligned}\mathbf{t}_r.\nabla \times (\boldsymbol{\pi}^{(1)*}_{(r)} - \boldsymbol{\tau}^{(1)*}_{(r)}) - \sum_q L_q \tilde{\nabla} S_q.\nabla \times (\boldsymbol{\pi}^{(1)*}_{(r)} - \boldsymbol{\tau}^{(1)*}_{(r)}) + \sum_q L_q \mathbf{t}_r.\nabla \times (\boldsymbol{\pi}^*_{(r)(q)} - \boldsymbol{\tau}^*_{(r)(q)})\\ + \sum_q L_q \mathbf{t}_r.\nabla \times [\tilde{\nabla} S_q.(\pi^{(1)} - \tau^{(1)})^*]\\ + \sum_q L_q \mathbf{t}_r.\nabla \times \{S_q(r\,\partial \boldsymbol{\pi}^{(1)}_{(r)}/\partial r - r\,\partial \boldsymbol{\tau}^{(1)}_{(r)}/\partial r)^*\} + O(L_q^2) = 0.\end{aligned}$$

From boundary condition (3*c*) we have that

$$\mathbf{t}_r.\nabla \times \boldsymbol{\pi}^{(1)*}_{(r)} = \mathbf{t}_r.\nabla \times \boldsymbol{\tau}^{(1)*}_{(r)},$$

and, since the coefficients L_q are independent, one readily obtains

$$\begin{aligned}\mathbf{t}_r.\nabla \times [\boldsymbol{\pi}^*_{(r)(q)} - \boldsymbol{\tau}^*_{(r)(q)}] = \tilde{\nabla} S_q.\nabla \times [\boldsymbol{\pi}^{(1)*}_{(r)} - \boldsymbol{\tau}^{(1)*}_{(r)}] + \mathbf{t}_r.\nabla \times \{\tilde{\nabla} S_q.[\pi^{(1)} - \tau^{(1)}]^*\}\\ - \mathbf{t}_r.\nabla \times \{S_q[(r\,\partial \boldsymbol{\pi}^{(1)}_{(r)}/\partial r) - (r\,\partial \boldsymbol{\tau}^{(1)}_{(r)}/\partial r)]^*\}.\end{aligned}$$

The following relation from Hetsroni & Haber (1970),

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{a}\left\{2 + \sum_{n=2}^{\infty} (n^2+n-2) L_n S_n\right\},$$

together with the boundary conditions of the first iteration yields

$$\boldsymbol{\pi}^{(1)*}_{(r)} - \boldsymbol{\tau}^{(1)*}_{(r)} = \mathbf{t}_r\left\{(\sigma/a)\left[2 + \sum_{n=2}^{\infty} (n^2+n-2) L_n S_n\right]\right\}.$$

Substitution in the previous equation results in

$$\begin{aligned}\mathbf{t}_r.\nabla \times [\boldsymbol{\pi}_{(r)(q)} - \boldsymbol{\tau}_{(r)(q)}]^* = \tilde{\nabla} S_q.\sum_{n=2}^{\infty} (n^2+n-2)(\sigma L_n/a)\nabla \times (S_n \mathbf{t}_r)\\ + \mathbf{t}_r.\nabla \times \{\tilde{\nabla} S_q.[\pi^{(1)} - \tau^{(1)}]^*\}\\ - \mathbf{t}_r.\nabla \times \{S_q[r\,\partial \boldsymbol{\pi}^{(1)}_{(r)}/\partial r - r\,\partial \boldsymbol{\tau}^{(1)}_{(r)}/\partial r]^*\}. \quad \text{(B 16)}\end{aligned}$$

The last boundary condition was written as (12*f*),

$$\mathbf{t}_n.\nabla \times \{\mathbf{t}_n \times [\boldsymbol{\pi}_{(n)}]_s\} = \mathbf{t}_n.\nabla \times \{\mathbf{t}_n \times [\boldsymbol{\tau}_{(n)}]_s\} + \mathbf{t}_n.\nabla \times \left\{\mathbf{t}_n \times \mathbf{t}_n \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\},$$

where the last term vanishes identically. Therefore, the boundary conditions can be written as follows:

$$\mathbf{t}_n.\nabla \times \{\mathbf{t}_n \times [\boldsymbol{\pi}_{(n)}]_s\} = \mathbf{t}_n.\nabla \times \{\mathbf{t}_n \times [\boldsymbol{\tau}_{(n)}]_s\}. \quad \text{(B 17)}$$

Expanding in series, using (B 15) and neglecting terms of $O(L_q^2)$ and higher, one obtains:

$$\begin{aligned}\mathbf{t}_n \times [\boldsymbol{\pi}_{(n)}]_s = \mathbf{t}_r \times \boldsymbol{\pi}^{(1)*}_{(r)} + \mathbf{t}_r \times \sum_q L_q \boldsymbol{\pi}^*_{(r)(q)} - \sum_q L_q \mathbf{t}_r \times \pi^{(1)*}.\tilde{\nabla} S_q\\ + \mathbf{t}_r \times [\sum_q L_q S_q (r\,\partial \boldsymbol{\pi}^{(1)}_{(r)}/\partial r)^*] - \sum_q L_q \tilde{\nabla} S_q \times \boldsymbol{\pi}^{(1)*}_{(r)}.\end{aligned}$$

Further manipulations result in

$$\mathbf{t}_n.\nabla\times\{\mathbf{t}_n\times[\boldsymbol{\pi}_{(n)}]_s\} = \mathbf{t}_r.\nabla\times[\mathbf{t}_r\times\boldsymbol{\pi}^{(1)*}_{(r)}] - \sum_q L_q\tilde{\nabla}S_q.\nabla\times[\mathbf{t}_r\times\boldsymbol{\pi}^{(1)*}_{(r)}]$$
$$+\sum_q L_q\mathbf{t}_r.\nabla\times[\mathbf{t}_r\times\boldsymbol{\pi}^{*}_{(r)(q)}] + \sum_q L_q\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times S_q[r\,\partial\boldsymbol{\pi}^{(1)}_{(r)}/\partial r]^*\}$$
$$-\sum_q L_q\mathbf{t}_r.\nabla\times(\mathbf{t}_r\times\pi^{(1)*}.\tilde{\nabla}S_q) - \sum_q L_q\mathbf{t}_r.\nabla\times(\tilde{\nabla}S_q\times\boldsymbol{\pi}^{(1)*}_{(r)}). \tag{B 18}$$

A similar expression is obtained for the internal normal stress vector $\boldsymbol{\tau}_{(n)}$, when τ replaces π in (B 18).

Substitution of (B 18) into (B 17), recalling that

$$\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times[\boldsymbol{\pi}^{(1)}_{(r)}-\boldsymbol{\tau}^{(1)}_{(r)}]^*\} = 0$$

and

$$\tilde{\nabla}S_q.\nabla\times\{\mathbf{t}_r\times[\boldsymbol{\pi}^{(1)}_{(r)}-\boldsymbol{\tau}^{(1)}_{(r)}]^*\} = 0,$$

yields, after some simplifications:

$$\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times[\boldsymbol{\pi}_{(r)(q)}-\boldsymbol{\tau}_{(r)(q)}]^*\} = -\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times S_q[r\,\partial\boldsymbol{\pi}^{(1)}_{(r)}/\partial r - r\,\partial\boldsymbol{\tau}^{(1)}_{(r)}/\partial r]^*\}$$
$$+\mathbf{t}_r.\nabla\times\{\mathbf{t}_r\times[\pi^{(1)}-\tau^{(1)}]^*.\nabla S_q\}$$
$$+\mathbf{t}_r.\nabla\times\left\{\tilde{\nabla}S_q\times\left[\mathbf{t}_r(\sigma/a)\left(2+\sum_{n=2}^{\infty}(n^2+n-2)\,L_nS_n\right)\right]\right\}. \tag{B 19}$$

Appendix C. Evaluation of coefficients in the boundary conditions

The boundary conditions of the second iteration were expressed in appendix B in terms of the known first-iteration solutions of the velocity fields. Now, we plan to express the boundary conditions in terms of the known coefficients of the unperturbed flow field, α_n, β_n and γ_n.

First, some preliminary calculations. Substituting (A 1) and (A 2) into (5a) one obtains the following:

$$\mathbf{u}^{(1)*} = \sum_{n=1}^{\infty}\{{}^1\gimel_n\gamma_n\tilde{\nabla}\times(\mathbf{t}_rS_n)+({}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n)\,\tilde{\nabla}S_n\}, \tag{C 1}$$

where the Hebrew letters Gimel, Beth and Aleph are defined as follows:

$$\left.\begin{aligned}{}^1\gimel_n &= (2n+1)/n(n+1)\,[\lambda(n-1)+n+2], \quad {}^1\aleph_n = (2n+3)/n(n+1)\,(1+\lambda),\\ {}^1\beth_n &= (2n-1)/n(n+1)\,(1+\lambda).\end{aligned}\right\} \tag{C 2}$$

Note that the coefficients ${}^1\aleph_n$, ${}^1\beth_n$ and ${}^1\gimel_n$ are merely functions of the integers n and of the ratio between the viscosities λ. Thus, the velocity in (C 1) is expressed in terms of surface harmonics S_n, the coefficients of the unperturbed field α_n, β_n and γ_n and the coefficients ${}^1\aleph_n$, ${}^1\beth_n$ and ${}^1\gimel_n$. We now proceed to express other useful terms on the same principle.

From (5a) and (A 1) and (A 2) and using Euler's theorem for homogeneous polynomials, one obtains

$$r\frac{\partial\mathbf{u}^{(1)}}{\partial r} = \sum_{n=1}^{\infty}\left\{n\nabla\times[\mathbf{r}\chi^{(1)}_n]+(n-1)\nabla\Phi^{(1)}_n+\frac{n+3}{2(2n+3)\mu_i}r^2\nabla p^{(1)}_n-\frac{n}{(2n+3)\mu_i}\mathbf{r}p^{(1)}_n\right\}, \tag{C 3}$$

and a similar expression from (4*a*) and (A 1) and (A 2) for $r(\partial \mathbf{v}^{(1)}/\partial r)$. Substitution and some simplification yield

$$\left[r\frac{\partial \mathbf{u}^{(1)}}{\partial r} - r\frac{\partial \mathbf{v}^{(1)}}{\partial r}\right]^* = \sum_{n=1}^{\infty} \{{}^2\beth_n \gamma_n \tilde{\nabla} \times (\mathbf{t}_r S_n) + ({}^2\aleph_n \alpha_n + {}^2\beth_n \beta_n)\tilde{\nabla} S_n\}, \tag{C 4}$$

where

$$\left.\begin{aligned} {}^2\daleth_n &= (1-\lambda)(n-1)(2n+1)/n(n+1)[\lambda(n-1)+(n+2)],\\ {}^2\aleph_n &= (1-\lambda)(2n+1)(2n+3)/(1+\lambda)n(n+1),\\ {}^2\beth_n &= (1-\lambda)(2n+1)(2n-1)/(1+\lambda)n(n+1). \end{aligned}\right\} \tag{C 5}$$

The radial stress vector inside the drop can be expressed as follows:

$$\boldsymbol{\tau}_{(r)}^{(1)} = \frac{\mu_i}{r}\sum_{n=1}^{\infty}\left\{(n-1)\nabla\times(\mathbf{r}\chi_n^{(1)}) + 2(n-1)\nabla\Phi_n^{(1)} - \frac{2n^2+4n+3}{(n+1)(2n+3)\mu_i}\mathbf{r}p_n^{(1)} + \frac{n(n+2)}{(n+1)(2n+3)\mu_i}r^2\nabla p_n^{(1)}\right\}, \tag{C 6}$$

from which, similar to (C 4), the following is obtained:

$$r\frac{\partial \boldsymbol{\tau}_{(r)}^{(1)}}{\partial r} = \frac{\mu_i}{r}\sum_{n=1}^{\infty}\left\{(n-1)^2\nabla\times(\mathbf{r}\chi_n^{(1)}) + 2(n-1)(n-2)\nabla\Phi_n^{(1)} - \frac{n(2n^2+4n+3)}{(n+1)(2n+3)\mu_i}\mathbf{r}p_n^{(1)} + \frac{n^2(n+2)}{(n+1)(2n+3)\mu_i}r^2\nabla p_n^{(1)}\right\}. \tag{C 7}$$

In an analogous procedure to (C 5), we get

$$\left[r\frac{\partial \boldsymbol{\pi}_{(r)}^{(1)}}{\partial r} - r\frac{\partial \boldsymbol{\tau}_{(r)}^{(1)}}{\partial r}\right]^* = \frac{\mu_e}{a}\sum_{n=1}^{\infty}\{{}^3\daleth_n\gamma_n\tilde{\nabla}\times(\mathbf{t}_r S_n) + ({}^3\aleph_n\alpha_n + {}^3\beth_n\beta_n)\tilde{\nabla}S_n + (\text{coeff})\,\mathbf{t}_r\}, \tag{C 8}$$

where

$$\left.\begin{aligned} {}^3\daleth_n &= (n-1)(n+2)(2n+1)(1-\lambda)/n(n+1)[\lambda(n-1)+n+2],\\ {}^3\aleph_n &= [(2n+3)/n(n+1)(1+\lambda)][2n(n+2)-\lambda(6n^2+4n-1)],\\ {}^3\beth_n &= [(2n-1)/n(n+1)(1+\lambda)][2(n^2-1)-\lambda(6n^2+8n+1)], \end{aligned}\right\} \tag{C 9}$$

and (coeff) is an undefined coefficient of the unit vector $\mathbf{t}_r$. This coefficient turns out to be insignificant, as shown in these calculations:

The well-known expression for the stress tensor $\boldsymbol{\tau}$ is

$$\boldsymbol{\tau} = -p_i\mathbf{I} + \mu_i[\nabla\mathbf{u} + (\nabla\mathbf{u})^T], \tag{C 10}$$

where $\mathbf{I}$ is the idem tensor. Using the known expressions for the components of the stress tensor, combined with the boundary conditions imposed on the first iteration, one obtains

$$(\boldsymbol{\pi}^{(1)} - \boldsymbol{\tau}^{(1)})^* = -(p_e^{(1)} - p_i^{(1)})\tilde{\mathbf{I}} + (\mu_e/a)(1-\lambda)[\tilde{\nabla}\mathbf{u}^{(1)*} + (\tilde{\nabla}\mathbf{u}^{(1)*})^T] + [\text{tensorial components of } \mathbf{t}_r\mathbf{t}_r, \mathbf{t}_r\mathbf{t}_\theta, \mathbf{t}_\theta\mathbf{t}_r, \mathbf{t}_\phi\mathbf{t}_r, \mathbf{t}_r\mathbf{t}_\phi], \tag{C 11}$$

where $\tilde{\mathbf{I}} = \mathbf{I} - \mathbf{t}_r\mathbf{t}_r$ and where the expression in the second brackets does not contribute to further computation.

Substituting Hetsroni & Haber's (1970) solution for the first iteration fields of $\mathbf{u}^{(1)}$, $p_i^{(1)}$ and $p_e^{(1)}$, i.e. (C 1), (5*b*) and (4*b*), into (C 11) one obtains

$$(\boldsymbol{\pi}^{(1)}-\boldsymbol{\tau}^{(1)})^* = \frac{\mu_e(1-\lambda)}{a}\sum_{n=1}^{\infty}\{{}^1\gimel_n\gamma_n[\tilde{\nabla}\tilde{\nabla}\times(\mathbf{t}_r S_n)+\tilde{\nabla}\tilde{\nabla}\times(\mathbf{t}_r S_n)^T]$$
$$+2({}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n)\tilde{\nabla}\tilde{\nabla}S_n\}+\tilde{\mathbf{I}}\frac{\mu_e}{a}\sum_{n=0}^{\infty}\left\{\frac{2n+3}{n(n+1)(1+\lambda)}[\lambda(4n^2+2n+1)-2(n+1)]\alpha_n\right.$$
$$\left.+\frac{2n-1}{n(n+1)(1+\lambda)}[\lambda(4n^2+6n+4)+2n]\beta_n\right\}S_n+\begin{pmatrix}\text{tensorial components of}\\ \mathbf{t}_r\mathbf{t}_r,\mathbf{t}_r\mathbf{t}_\theta,\mathbf{t}_\theta\mathbf{t}_r,\mathbf{t}_r\mathbf{t}_\phi,\mathbf{t}_\phi\mathbf{t}_r\end{pmatrix}. \tag{C 12}$$

Substituting (C 1), (C 4), (C 8) and (C 12) into the transformed boundary conditions of the second iteration, one obtains, after some laborious calculations, the following expressions:

$$\left.\begin{aligned}
a_i^{q,n} &= {}^1\gimel_n\gamma_n f_i^{q,n}+({}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n)[h_i^{q,n}-n(n+1)g_i^{q,n}],\\
b_i^{q,n} &= -{}^2\gimel_n\gamma_n f_i^{q,n}-({}^2\aleph_n\alpha_n+{}^2\beth_n\beta_n)[h_i^{q,n}-n(n+1)g_i^{q,n}],\\
c_i^{q,n} &= {}^2\gimel_n\gamma_n[n(n+1)g_i^{q,n}-h_i^{q,n}]+({}^2\aleph_n\alpha_n+{}^2\beth_n\beta_n)f_i^{q,n},\\
d_i^{q,n} &= (1-\lambda)\,{}^1\gimel_n\gamma_n j_i^{q,n}+2(1-\lambda)({}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n)k_i^{q,n}\\
&\quad+2n(n+1)(1-\lambda)[{}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n]f_i^{q,n}-{}^3\gimel_n\gamma_n[n(n+1)g_i^{q,n}-h_i^{q,n}]\\
&\quad-({}^3\aleph_n\alpha_n+{}^3\beth_n\beta_n)f_i^{q,n},\\
e_i^{q,n} &= -{}^3\gimel_n\gamma_n f_i^{q,n}-({}^3\aleph_n\alpha_n+{}^3\beth_n\beta_n)[h_i^{q,n}-n(n+1)g_i^{q,n}]\\
&\quad+(1-\lambda)\,{}^1\gimel_n\gamma_n m_i^{q,n}+2(1-\lambda)({}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n)l_i^{q,n}\\
&\quad-2(1-\lambda)(n+1)[{}^1\aleph_n\alpha_n+{}^1\beth_n\beta_n].[h_i^{q,n}-q(q+1)g_i^{q,n}],
\end{aligned}\right\} \tag{C 13}$$

where $f_i^{q,n}$, $g_i^{q,n}$, $h_i^{q,n}$, $k_i^{q,n}$, $l_i^{q,n}$ and $j_i^{q,n}$ are defined by the following equation:

$$\left.\begin{aligned}
S_qS_n &= \sum_{i=0\text{ or }1}^{n+q} g_i^{q,n}S_i,\\
\tilde{\nabla}S_q.\tilde{\nabla}S_n &= \sum_{i=0\text{ or }1}^{n+q} h_i^{q,n}S_i,\\
\mathbf{t}_r.\tilde{\nabla}S_q\times\tilde{\nabla}S_n &= \sum_{i=0\text{ or }1}^{n+q-1} f_i^{q,n}S_i,\\
\mathbf{t}_r.\tilde{\nabla}\times(\tilde{\nabla}S_q.\tilde{\nabla}\tilde{\nabla}S_n) &= \sum_{i=0\text{ or }1}^{n+q-1} k_i^{q,n}S_i,\\
\tilde{\nabla}.(\tilde{\nabla}S_q.\tilde{\nabla}\tilde{\nabla}S_n) &= \sum_{i=0\text{ or }1}^{n+q} l_i^{q,n}S_i,\\
\tilde{\nabla}.\{\tilde{\nabla}S_q.[(\tilde{\nabla}\tilde{\nabla}\times\mathbf{t}_rS_n)+(\tilde{\nabla}\tilde{\nabla}\times\mathbf{t}_rS_n)^T]\} &= \sum_{i=0\text{ or }1}^{n+q-1} m_i^{q,n}S_i,\\
\mathbf{t}_r.\tilde{\nabla}\times\{\nabla S_q.[(\tilde{\nabla}\tilde{\nabla}\times\mathbf{t}_rS_n)+(\tilde{\nabla}\tilde{\nabla}\times\mathbf{t}_rS_n)^T]\} &= \sum_{i=0\text{ or }1}^{n+q} j_i^{q,n}S_i.
\end{aligned}\right\} \tag{C 14}$$

Here too we recall that S_q, for example, is an abbreviation for $2q+1$ expressions of the form

$$\begin{matrix}\sin\\ \cos\end{matrix}\, m\phi P_q^m \quad (m = 0, 1, \ldots, q).$$

Hence, the product $S_n S_q$ symbolizes the product

$$\left.\begin{matrix}\sin \\ \cos\end{matrix} m\phi P_q^m \begin{matrix}\sin \\ \cos\end{matrix} p\phi P_n^p \right\} \quad \left\{\begin{matrix}(m = 0, 1, \ldots, q); \\ (p = 0, 1, \ldots, n);\end{matrix}\right.$$

thus, the coefficients $g_i^{q,n}$, $h_i^{q,n}$, etc. are functions of i, g, n as well as the indices p and n.

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